Irregular Sampling in Satellite Images and Reconstruction Algorithms

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Abstract

Satellite images are sampled on a slightly perturbed grid, due to micro-vibrations of the instrument during capture. This perturbation can be estimated with high accuracy, but it must be also corrected in the images for certain stereo and multi-spectral applications. In this work we show to what an extent future satellites being developed at CNES satisfy the conditions required by irregular sampling theory to make the problem of resampling on a regular grid well posed, whereas most current imaging systems do not allow for such a well posed reconstruction due to aliasing. Then we discuss the available reconstruction algorithms and propose a new one, which is to some extent better adapted to the sampling conditions of such satellites.

1 Introduction

Satellite images are not sampled on an exactly regular grid, but rather on a slightly perturbed grid. The sources of these perturbations include: micro-vibrations of the satellite while it takes the image, and irregularities in the position of the sensors on the image plane. For certain satellite images, the combined effect of these perturbations can be automatically estimated for each image, by different means developed at CNES. Physical models of the satellites exists, which allow to predict certain vibration modes that can be activated at one time or another. For this reason the perturbations in the sampling grid can be modeled quite accurately by:

\[
\varepsilon(x) = \sum_{k=1}^{M} a_k(x) \cos(\omega_k x + \varphi_k)
\]

where the modulation functions \(a_k(x)\) are extremely smooth and bounded by a constant \(A_k\) which is usually inversely proportional to the frequency \(\omega_k\). The vibration frequencies \(\omega_k\) themselves are usually at least an order of magnitude smaller than the Nyquist cut-off frequency associated to the sampling rate. The number \(M\) of vibration modes is also relatively small. This produces very smooth perturbations where the distance between successive samples does not deviate from one pixel by more than about 10%. But accumulated over a large distance the absolute perturbation may reach several pixels. All in all the perturbation is so smooth that it does not produce a visible geometrical distortion of the image, so its correction is not strictly necessary from an aesthetic or presentation point of view. However, such perturbations must be taken into account by algorithms which interpolate these images to obtain sub-pixel accuracy. Such an example is the production of highly accurate digital elevation models from stereo pairs [19], or super-resolution of color images from multi-spectral images using a high resolution panchromatic...
channel [27]. These applications require image registration with an accuracy even finer than 0.1 pixels in the disparity map. To achieve such an accuracy, the micro-vibrations in the original image sampling must be corrected before registration.

In this work we study the problem of resampling the image on a regular grid, given its samples on a perturbed grid and the corresponding perturbation. We note that this perturbation can be obtained with a high level of accuracy from cues given both by gyroscopes mounted on the satellite and by analyzing the images themselves [25]. Nevertheless we shall not deal here with the estimation of the perturbation (i.e. the position of the sampling points in the irregular grid), and we shall rather assume that the irregular grid is given with a high level of accuracy.

The article is organized as follows. First (section 2) we review the conditions required by irregular sampling theory to make the reconstruction problem well-posed, and we analyze how these conditions apply to satellite imaging systems. Then (section 3) we review some state-of-the-art reconstruction algorithms and point out the characteristics of satellite images that are not exploited by these methods. Next (section 4), we propose a new algorithm, based on a pseudo-inverse iteration, which better exploits these characteristics. Finally, we discuss the results of our simulations (section 5).

2 Existence theorems

2.1 Problem statement.

We assume that the continuous image \( f \) before sampling is band-limited, i.e. that \( f \in L^2(\mathbb{R}^2) \) and \( \text{supp}(\hat{f}) \subseteq [-\pi, \pi]^2 \). We shall note this space of band-limited functions by \( PW \), for Payley-Wiener. As it has been observed in [5] this is not strictly the case in real systems, and that sometimes the reciprocal cell needs to be adapted to the transfer function. Furthermore, even with an adapted reciprocal cell, \( \hat{f} \) may have an important part of the energy outside this cell.

We shall return later in this section to the question of to what an extent sampling systems satisfy these hypotheses. For the moment we keep it because most of the theory on irregular sampling is developed in this framework, and results are not straightforwardly generalizable to arbitrary spectral domains.

Secondly, we assume that we know the positions of the samples

\[ \Lambda = \{ \lambda_k \}_{k \in \mathbb{Z}^2}, \quad \lambda_k \in \mathbb{R}^2 \]  

and the values of \( f \) at these points

\[ \tilde{s}_k = f(\lambda_k), \quad k \in \mathbb{Z}^2. \]  

Then the problem consists of finding the regular samples of \( f \) from the vector \( \tilde{s} \) of irregular samples

\[ s_k = f(k), \quad k \in \mathbb{Z}^2. \]  

In the general irregular sampling case, the sampling set \( \Lambda \) doesn’t need to have any particular structure (beyond a minimal density), and some theorems and algorithms apply to this general case. In satellite imaging systems, however, we shall restrict ourselves to the perturbed sampling case, in which the samples

\[ \lambda_k = k + \varepsilon(k), \quad k \in \mathbb{Z}^2 \]  

are a perturbation of the regular grid, where some additional properties may be assumed about the particular form of the perturbation function \( \varepsilon \). Classical results make statements about the amplitude of \( \varepsilon \), even though a model of the form given in equation (1) would be more appropriate in satellite imaging. Here we shall emphasize the fact that \( \varepsilon \) may be assumed...
regular (low frequency) with respect to the image $f$.

Before we present the existence results we should make a precision. We are only interested in stable reconstructions. The precise sense of stable reconstructions is slightly different in different settings, but in general terms, it intends to make sure that the operator mapping $f(\Lambda)$ to $f(\mathbb{Z}^2)$ (or equivalently to $f$) has a bounded norm, not depending on $f \in PW$.

We divide the remaining of this section in two parts. First we study the perturbed sampling case, where precise bounds on the amplitude of $\varepsilon$ are known if we are sampled at the Nyquist rate (section 2.2). Then we discuss the results that are known in the general irregular sampling case (section 2.3). These statements have two differences with respect to the results in section 2.2: First they require over-sampling (in the generalized sense of Beurling-Landau densities); Secondly, when interpreted in terms of perturbed sampling, these results can be rewritten in terms of the regularity of the grid perturbation $\varepsilon$, instead of its amplitude.

### 2.2 Perturbed critical sampling

The main result on the limits of perturbed sampling in the one-dimensional case is due to Kadec [21] (see also [29, section 1.10, page 42]) and can be stated as follows:

**Theorem 1 (1D Kadec).** If there is a constant $c$ such that

$$|\varepsilon(k)| = |\lambda_k - k| \leq c < \frac{1}{4}$$

(6)

then the family $\{e^{i\lambda_k \xi}\}_{k \in \mathbb{Z}}$ forms a Riesz basis of $L^2[-\pi, \pi]$. Hence there exists a stable reconstruction formula of any band-limited function $f \in L^2(\mathbb{R})$ from its irregular samples $f(\lambda_k)$.

Note that in this case stability of the reconstruction is formulated in terms of existence of a Riesz basis in the Fourier domain. The fact that the functions $\varphi_k(\xi) = e^{i\lambda_k \xi}$ form a Riesz basis implies that the operator

$$G : L^2[-\pi, \pi] \rightarrow L^2[-\pi, \pi] \quad G(f) = \sum_k \langle f, \varphi_k \rangle \varphi_k$$

(7)

is invertible, auto-adjoint and positive. Then the family $\{\tilde{\varphi}_k\}$ defined by $\tilde{\varphi}_k = G^{-1}(\varphi_k)$ is also a Riesz basis biorthogonal to the $\{e^{i\lambda_k \xi}\}_{k \in \mathbb{Z}}$. Finally the reconstruction formula is provided by

$$\hat{f}(\xi) = \sum_k \langle \hat{f}, \varphi_k \rangle \tilde{\varphi}_k(\xi)$$

(8)

$$= \sum_k f(\lambda_k)\tilde{\varphi}_k(\xi) = \sum_k f(\lambda_k)G^{-1}\varphi_k(\xi)$$

(9)

$$= G^{-1}\sum_k f(\lambda_k)\varphi_k(\xi),$$

(10)

where $G^{-1}$ may be computed iteratively by a Neumann series, because $G = a(I - R)$, with $a > 0$ and $\|R\| < 1$. However this iteration may be too slow to converge and more efficient numerical methods are required.

Another important observation about Kadec’s theorem is that the maximal value of the constant for the theorem to hold is known exactly. In fact we can build counterexamples with $\varepsilon(x) \leq \frac{1}{4}$, for which the theorem doesn’t hold (see [29, p 44 and section 3.3 p 122]). This is not the case for the generalization to two dimensions:

**Theorem 2 (2D Kadec).** If the perturbation $\varepsilon$ is such that

$$|\varepsilon(k)| = \|\lambda_k - k\| \leq 0.11$$

(11)

then the family $\{e^{i(\lambda_k \xi)}\}_{k \in \mathbb{Z}^2}$ forms a Riesz basis of $L^2([-\pi, \pi]^2)$. Hence there exists a stable reconstruction formula of any band-limited function $f \in L^2(\mathbb{R}^2)$ from its irregular samples $f(\lambda_k)$.
The first generalization of this kind is due to Favier and Zalik [11], where they give a bound of 0.05, and this bound was later improved by Chui and Shi [9]. Nevertheless the bound is not shown to be sharp as in the one-dimensional case.

For a particular kind of perturbations that we call separable Jaffard [20] showed that the two-dimensional case can be reduced to the one-dimensional case, and therefore the 0.25 constant is still valid.

**Theorem 3 (separable 2D Kadec).** Assume that the perturbation $\varepsilon$ is separable, i.e. the perturbation

$$\varepsilon(n, m) = \left( \delta_n^1, \delta_n^2 \right) = \begin{pmatrix} 0 \\ \theta_m^2 \end{pmatrix}$$

(12)

can be expressed as a combination of an arbitrary vector only depending on the horizontal coordinate $n$, plus a vertical correction only depending on the vertical coordinate $m$. If in addition there is a constant $c$ such that

$$|\delta_n^1| \leq c \quad \text{and} \quad |\theta_m^2| \leq c \quad \text{and} \quad c < \frac{1}{4} \quad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}$$

(13)

then the family \( \{ e^{i(n\lambda_1, \xi_1)} \}_{n,m} \subset \mathbb{Z}^2 \) forms a Riesz basis of \( L^2([-\pi, \pi]^2) \).

Observe in particular that there is no condition on \( \delta_n^2 \).

The proof proceeds by showing that the Riesz basis biorthogonal to \( \{ e^{i(n\lambda_1, \xi_1)} \}_{n,m} \subset \mathbb{Z}^2 \) can be written as \( \{ \bar{\varphi}_n(\xi_1) \bar{\psi}_m(\xi_2) e^{i\delta_n^1 \xi_1 + i\delta_n^2 \xi_2} \}_{n,m} \subset \mathbb{Z}^2 \) where \( \{ \bar{\varphi}_n(\xi_1) \}_{n \in \mathbb{Z}} \) and \( \{ \bar{\psi}_m(\xi_2) \}_{m \in \mathbb{Z}} \) are (one-dimensional) Riesz bases biorthogonal to \( \{ e^{i(n \lambda_1, \xi_1)} \}_{n \in \mathbb{Z}} \) and \( \{ e^{i(m \lambda_2 + \xi_2)} \}_{m \in \mathbb{Z}} \) respectively. Hence the result follows by applying twice the one-dimensional Kadec theorem.

Jaffard’s result is of interest in satellite imaging because many of the sampling systems analyzed in [5] and [4, chapter 1] actually impose that the perturbations be separable if we choose the coordinate system in a convenient way (see [4, annex B] for more details). Changing the coordinate system also changes the shape of the canonical reciprocal cell \( [-\pi, \pi]^2 \). In some cases this change goes in the correct sense, making \( f \) more close to band-limited, in other cases the converse is true. In table 1, we summarize the results of applying both versions of the two-dimensional Kadec theorem to some of the sampling systems in [5]. In some cases the band-limited hypothesis becomes “less true” in other cases the bound on the maximal perturbations that are allowed becomes smaller. Clearly the Hypermode system yields the best results, much better than the Sq2 system which offers roughly the same effective resolution. The reader should be warned to take this comparison with care, since the table only represents the best combination of perturbation bounds and closeness to band-limited that we were able to show with the mathematical results that are available. However, these bounds are by no means necessarily the optimal ones.

### 2.3 Irregular over-critical sampling

When we consider general irregular sampling sets (not necessarily perturbations of the regular grid), the stable reconstruction condition is usually stated as follows.

**Definition 1 (set of (stable) sampling).** A set \( \Lambda = \{ \lambda_k \}_{k \in \mathbb{Z}^d} \) is called a set of stable sampling or simply set of sampling if there is a constant \( C > 0 \) such that

$$\| f \|_{L^2} \leq C \| f(\Lambda) \|_{L^2}$$

(14)

holds for all band-limited functions \( f \in PW \).
Deviation from band-limitedness

|            | $D^*_{\text{non-sep}}$ | $D^*_{\text{sep}}$ | sup $|\varepsilon|$ |
|------------|------------------------|--------------------|----------------------|
| Hipermode  | 1.7%                   | 1.7%               | 0.25p = 0.13c        |
| Sq0        | 18.0%                  | 18.0%              | 0.25p = 0.25c        |
| Sq1        | 6.0%                   | 13.0%              | 0.11p = 0.08c        |
| Sq2        | 2.2%                   | 13.2%              | 0.11p = 0.05c        |
| Hex0       | 16.3%                  | 16.3%              | 0.25p = 0.25c        |
| Hex1       | 2.2%                   | 12.7%              | 0.11p = 0.06c        |
| Hex2       | 0.6%                   | 13.5%              | 0.11p = 0.04c        |

Table 1: Comparison of several systems in terms of “band-limitedness” and maximal perturbations allowed for stable reconstruction. $D^*_{\text{non-sep}}$ represents the canonical reciprocal cell $[-\pi, \pi]^2$, in which $\hat{f}$ has to be band-limited to apply the non-separable 2D Kadec theorem 2. $D^*_{\text{sep}}$ represents the canonical reciprocal cells $[-\pi, \pi]^2$ in which $\hat{f}$ has to be band-limited to apply the separable 2D Kadec theorem 3. This reciprocal cell is obtained after a suitable change of coordinate system as explained in annex B, section 5. In the separable case we can apply the result of Jaffard with maximal perturbation 0.25 pixels, otherwise we can only use the result of Chui with maximal perturbation 0.11 pixels. (Remember from [5] and [4, chapter 1] that this distance in pixels represents different values for the different sampling systems if expressed as a multiple of the size $c$ of the sensors, which represents a constant distance on the ground.) However, for some changes of coordinate system the image becomes very far away from band-limited. Here we measure the deviation from band-limitedness as the percentage $100 \sqrt{\int_{\mathbb{R}^2 \setminus D^*} |H|^2 / \int_{D^*} |H|^2}$ of the transfer function $H$ that lies outside of the chosen spectral domain $D^*$. We observe that except for Hypermode, Sq0 and Hex0, the best band-limitedness occurs in the non-separable case. (However Sq0 and Hex0 are not sufficiently band-limited anyway). So we can apply the separable 2D Kadec theorem, and still obtain the smallest possible deviation from band-limitedness, only for Hypermode, Sq0 and Hex0.
A classical result by Beurling and Landau [6, 22], states necessary and sufficient conditions for stable sampling in terms of the Beurling density. A simplified version of their results can be written as follows:

\[ D(\Lambda) = \lim_{r \to 1} \inf_{x \in \mathbb{R}^d} \frac{\#(B_r(x) \cap \Lambda)}{(r)^d} \quad (15) \]

where \( B_r(x) \) is a cube of side \( r \) and centered at \( x \).

**Theorem 4 (Beurling-Landau).** If \( \Lambda \) is a set of stable sampling then \( D(\Lambda) \geq 1 \). Conversely, if \( D(\Lambda) > 1 \) then \( \Lambda \) is a set of stable sampling.

Even if the result was shown originally in the one-dimensional Payley-Wiener framework, several generalizations have been proposed to higher dimensions and shift-invariant \( L^p \) spaces [17, 2]. See also [3, p 6] for a review of different generalizations of this result.

**Remark 1.** The definition of set of sampling should be distinguished from the weaker condition of set of uniqueness which simply requires that for all band-limited functions \( f \), if \( f(\Lambda) = 0 \) then \( f = 0 \). It can be shown that there are sets of uniqueness with arbitrarily small density whereas sets of sampling satisfy \( D(\Lambda) \geq 1 \).

**Remark 2.** Note that Kadec’s theorem deals with the critically sampled case where \( D(\Lambda) = 1 \) which is undetermined by Beurling-Landau’s results. On the other hand Beurling’s result provides a sufficient condition for stable reconstruction in cases where Kadec’s theorem does not apply (because, e.g. the perturbation is too big) but requires oversampling \( (D(\Lambda) > 1) \) for the sufficient condition to hold.

In practice, even if the Beurling density is larger than 1, the ratio between the frame constants may be extremely large, thus making the problem very ill conditioned. In order to obtain well conditioned systems, practical reconstruction algorithms must impose a stronger condition, that involves a more local version of Beurling’s density, namely

**Definition 2.** A sampling set \( X \subseteq \Omega \) is said to be \( \gamma \)-dense if

\[ \bigcup_{x \in X} B_\gamma(x) = \Omega. \quad (16) \]

Note that \( \gamma \) is, strictly speaking, the inverse of a density, in the sense that in the \( d \)-dimensional case

\[ D(\Lambda) \geq \frac{1}{\gamma^d} \]

where equality holds e.g. if the sampling set \( \Lambda \) is a regular grid and \( \gamma \) is the minimal radius such that \( \Lambda \) is \( \gamma \)-dense. Hence, to make the link to [5] and [4, chapter 1], \( \gamma \) plays the role of spatial resolution \( R_{\text{nom}} \), whereas \( D(\Lambda) \) plays the role of spectral resolution \( r_{\text{nom}} \). In section 3.2 we shall review two algorithms which are shown to be convergent when the sampling set is \( \gamma \)-dense, with \( \gamma < 1 \) (and thus \( D(\Lambda) > 1 \)). Furthermore Gröchenig [16] proposed an extension of this proof of convergence, which could be used to give a new proof of the Beurling-Landau result, “with the additional benefit of an explicit and efficient numerical algorithm attached to the proof” (quoted from [16]).

### 3 Available Reconstruction Algorithms

#### 3.1 Based on Kadec’s condition (perturbed sampling)

There are a number of works that propose numerical algorithms that are convergent under the conditions of Kadec’s theorem [13]. We did not consider such algorithms in this work for two
reasons: (i) They are formulated in the one-dimensional case, and their generalization to the two-dimensional case is not self-evident, except possibly in the separable case; (ii) They require either high oversampling rates or very small perturbation amplitudes for the filters to be of a reasonable size when the method is generalized to more dimensions. For these reasons they did not seem to be of interest in our particular case.

3.2 Based on Beurling-Landau type conditions (irregular sampling)

3.2.1 ACT algorithm

One of the best performing reconstruction methods available is the ACT algorithm developed initially by Feichtinger, Gröchenig and Strohmer [12] and further analyzed, refined and generalized by Gröchenig, Strohmer and Rauth [18, 24]. The method intelligently combines an accelerated version of the frame iteration derived from the proof of Kadec’s theorem, with adaptive weights in order to improve the condition number of the problem, a conjugate gradient iteration which accelerates convergence, and the formulation of the problem as a Toeplitz system which makes the computation of each iteration even faster. Furthermore the preparation steps before the conjugate gradient iteration can start, can benefit from the USFFT (for unequally spaced fast Fourier transform) algorithm by Beylkin [7, 8].

More precisely, the algorithm is based on a representation of an \( N \times N \) periodic band-limited function \( f \) as a trigonometric polynomial of order \( N^2 \)

\[
f(x) = \sum_{n \in [1,N]^2} a_n e^{2\pi i n^T (x)}
\]

so that the interpolation conditions become

\[
\mathbf{s}_k = f(\lambda_k) = \sum_{n \in [1,N]^2} a_n e^{2\pi i n^T (\lambda_k)}
\]

or equivalently in matrix form

\[
\mathbf{s} = V \mathbf{a}, \quad \text{where} \quad V = ((v_{kn})), \quad v_{kn} = e^{2\pi i n^T (\lambda_k)}
\]

i.e. \( V \) is the van der Monde matrix associated to the trigonometric polynomial. Now the problem is reduced to solving for \( \mathbf{a} \) the system of linear equations (19). But if \( \Lambda \) contains some regions with extremely dense sampling the system will not be well balanced. In order to improve the condition number we multiply the \( k \)-th equation by a weight

\[
w_k = \text{area} \left( \{ x : |x - \lambda_k| < |x - \lambda_j|, \forall j \neq k \} \right)
\]

which is inversely proportional to the sampling density at \( \lambda_k \). In addition, instead of solving the linear system (19) directly, it will be more convenient to solve the normal equations as an optimization problem

\[
\min_{\mathbf{a}} \| V^* \text{diag}(w) V \mathbf{a} - V^* \text{diag}(w) \mathbf{s} \|^2
\]

because the \( N^2 \times N^2 \) matrix \( T = V^* \text{diag}(w) V \) can be shown to have Toeplitz structure, so that the multiplication \( Ta \) can be efficiently computed in \( N^2 \log_2(N^2) \) time using Fourier methods.

In addition the non-harmonic series

\[
t_n = \sum_{k \in [1,M]^2} e^{-2\pi i n^T (\lambda_k)} w_k
\]

\[
b_n = \sum_{k \in [1,M]^2} e^{-2\pi i n^T (\lambda_k)} w_k \mathbf{s}_k
\]
defining $T = V^* \text{diag}(w)V$ and $b = V^* \text{diag}(w)\tilde{s}$ in equation (21) can be approximated using the USFFT [7] in $CM^2 \log_2(M^2)$ time each, where $C$ is a constant, which is inversely proportional to the required precision.

Algorithm 1 Reconstruction from irregular samples using ACT algorithm

Require: $M^2$ irregular samples in vector $\tilde{s}$, and degree $N^2 \leq M^2$ of trigonometric polynomial.
Ensure: $N^2$ regular samples in vector $s$.
1: Compute $T = V^* \text{diag}(w)V$ and $b = V^* \text{diag}(w)\tilde{s}$ using the USFFT
2: Minimize $\|Ta - b\|^2$ using conjugate gradients.
3: Compute the regular samples $s_k = f(k)$ for $k \in [1, N]^2$ by applying the inverse FFT to $a$.

The overall procedure can be summarized as shown in algorithm 1. The complexity of step 1 is as we said $O(2M^2 \log(M^2))$, the second step takes $4N^2 \log_2(N^2)$ per iteration. The conjugate gradient algorithm ensures convergence (under certain conditions) in $N^2$ iterations, but numerically, there is little precision to gain beyond about $N$ iterations. Finally step 3 is computed very fast in $N^2 \log_2(N^2)$ iterations. Note that the method is particularly efficient for highly oversampled data, i.e. $N \ll M$, for two reasons: (i) if we are highly oversampled, as we shall see later the convergence rate of CG is faster; (ii) The bulk of the work depending on the large number $M^2$ of samples is done in step 1. The rest of the algorithm’s complexity depends only on the size $N^2$ of the spectrum. In our case, however, we are critically sampled ($M = N$) and the total complexity becomes about $4N^3 \log_2(N^2)$, if CG converges in $N$ iterations. In [18, 24] many criteria are given, to obtain estimates of error at each iteration, to decide when to stop the algorithm (because no further improvement of the approximation is possible), as well as to estimate the optimal size $N^2$ of the spectrum.

The convergence rate of the CG algorithm is determined by the condition number $\kappa = \text{cond}(T)$, or the ratio of the largest to the smallest eigenvalue of $T$. More precisely at each iteration the approximation error is decreased by a factor $\frac{\kappa - 1}{\kappa + 1}$ [14]. Gröchenig and Strohmer [15, 18] provided a useful characterization of the condition number of $T$ in the 2-dimensional case

Proposition 1 (ACT convergence rate). If the sampling set is $\gamma$-dense with

$$\gamma < \frac{\log 2}{4\pi}$$

then the condition number of $T$ is

$$\kappa \leq \frac{4}{(2 - e^{4\pi \gamma})^2}$$

and algorithm 1 converges to the exact solution $s$.

Note that the sampling has to be much more dense than the critical sampling rate, for the algorithm to ensure convergence to the exact result, which is not the case in satellite imaging. Nevertheless, even when $T$ is not invertible (and has an infinite condition number), the CG iteration chooses among the minimizers of $\|Ta - b\|$ the one of minimal norm, i.e. $a = T^+b$, where $T^+$ is the pseudo-inverse of $T$. Therefore, if we know the a-priori spectral decay rate of the image

$$|\hat{f}(\omega)| \leq C\phi(\omega)$$

typically $\phi(\omega) = (1 + \omega)^{-r}$ for $r \in [1, 2]$ or a combination of this natural decay rate, and the known transfer function, in that case we can regularize the solution by imposing this decay rate. Specifically we minimize the modified system

$$\min \|TDc - b\|, \quad a = Dc$$

The bound on the density $\gamma$ can be significantly relaxed in the one-dimensional case and in the case of a “separable” two-dimensional perturbation.

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where $D = \text{diag}(\{\phi(\frac{2\pi}{N}n)\}_n)$ is a diagonal matrix containing the corresponding values of $\phi(\omega)$. This way we shall obtain the solution solution $a = D(TD)^{+} b$, which is among all minimizers of $\|Ta - b\|$ the one which best follows the prescribed spectral decay rate, i.e. $D^{-1} a$ is of minimal norm.

This discussion explains why the ACT algorithm of Gröchenig and Strohmer [18] provides good approximations to the exact solution, even when the convergence conditions in proposition 1 are not satisfied as in the case of satellite imaging, while at the same time the convergence rate is not that good.

We also tried limiting the size $N$ of the spectrum to a smaller value than $M$, in such a way that the density condition in proposition 1 is satisfied. In this case we cannot expect to recover the high frequencies of $f$, but convergence should be faster. In practice, the accelerated convergence is minimal and does not compensate for the loss in accuracy, so we always kept the regularized version described above, imposing a polynomial decay for $\hat{f}$.

### 3.2.2 Aldroubi’s algorithm

More recently (actually we discovered this method by the end of this project) Aldroubi and Gröchenig [3] proposed a different method which has different convergence properties, and is directly formulated in the more general shift-invariant weighted $d$-dimensional $L^p$ spaces $V_{\nu}^p(\phi)$, where $\nu$ is a weight that is introduced in the $L^p$ norm to control the decay rate of the function in the spatial domain, and the generator $\phi$ must satisfy certain regularity properties that depend on $p$ and $\nu$. The particular Payley-Wiener framework we are dealing with here is obtained for $p = 2$, $\nu = 1$ and $\phi = \text{sinc}$.

Assume that our sampling set $S$ is $\gamma_0$-dense, so that $\mathbb{R}^d = \bigcup_{k} B_\gamma(\lambda_k)$ for any $\gamma > \gamma_0$. Now build a partition of unity $\{\beta_k\}_k$ such that

$$\begin{align*}
0 & \leq \beta_k \leq 1 \\
\text{supp}(\beta_k) & \subset B_\gamma(\lambda_k) \\
\sum_k \beta_k & = 1.
\end{align*}$$

Such a partition is called a bounded partition of unity. Then define an operator $Q_\Lambda$ by:

$$Q_\Lambda f = \sum_k f(\lambda_k) \beta_k.$$  \hfill (27)

Note that for the particular partition of unity given by $\beta_k = \nu_k$, where $V_k$ is the Voronoi cell associated to $\lambda_k$ in the set $\Lambda$, the quasi-interpolant operator $Q_\Lambda$ corresponds to nearest neighbour interpolation. This construction, however, allows for a more smooth and bounded interpolation in the case where we take for instance irregular spline functions of higher order.

Furthermore we consider the projection operator $P$ from $L^p$ to $V_{\nu}^p(\phi)$ (in the Payley-Wiener framework this corresponds to removing the frequencies beyond $[-\pi, \pi]^2$), then for sufficiently small $\gamma$ it is shown that the operator $PQ_\Lambda - I$ is a contraction. This justifies the iteration

$$\begin{align*}
\begin{cases}
f_1 & = PQ_\Lambda f \\
 f_{n+1} & = PQ_\Lambda (f - f_n) + f_n = f_1 + (PQ_\Lambda - I) f_n
\end{cases}
\end{align*}$$  \hfill (28)

with convergence

$$\|f - f_n\|_{L^p} \leq C \alpha^n \|f - f_1\|_{L^p}$$  \hfill (29)

for some convergence rate $\alpha < 1$ depending on $\gamma$.

Now we explain how we implemented the operators $PQ_\Lambda$ and $(PQ_\Lambda - I)$ in the Payley-Wiener framework, since this is not explained in detail in [3]. At each step we assume that $f_n$ is
represented by its samples on $\mathbb{Z}^2$. This is sufficient to completely reconstruct $f_n(x)$ at any point $x \in \mathbb{R}^2$, because – thanks to the projector $P$ – we have $f_n \in PW$. On the other hand, the result of applying the quasi interpolator $Q_A f_n$ or $Q_A f$ must be computed on a finer grid $(\mathbb{Z}/z)^2$ for some zoom factor $z > 1$, otherwise there would be no sense in applying the projector $P$ afterwards. The choice of this zoom factor is for the moment quite arbitrary, in the absence of a criterion that allows us to compute its influence in the approximation error.\footnote{The choice of $z$ has no reason to be band-limited, nor in any spline-like space, even if $f$ is band-limited. However, the Fourier coefficients of $Q_A f$ should exhibit a certain decay rate of its Fourier coefficients, which depends on the sampling density. From this decay rate it should be possible to compute a more accurate criterion to choose the appropriate zoom factor $z$ for a given precision.} As a rule we took $z = n$ when a precision of $10^{-n}$ was required. This choice seemed to be enough in our experiments (larger values of $z$ did not significantly improve the convergence rate), but it is possibly excessive.

In order to save time in the calculation of the nearest-neighbour interpolation, we pre-compute at the beginning of the iteration the mapping $x \mapsto k(x)$ which assigns to each point $x \in (\mathbb{Z}/z)^2$ in the fine grid, the index $k$ to the closest point $\lambda_{k(x)} \in \Lambda$ in the irregular sampling grid. Then the quasi-interpolator is computed for all $x \in (\mathbb{Z}/z)^2$ simply by

\begin{align}
(Q_A f)(x) &= \tilde{s}_k(x) \\
(Q_A f_n)(x) &= f_n(\lambda_{k(x)})
\end{align}

which shows the need of resampling $f_n(\Lambda)$ on the irregular grid from its regular samples $f(\mathbb{Z}^2)$. This is an easier problem than the inverse irregular to regular sampling problem we are trying to solve, since we only need to do Shannon interpolation. However, we cannot do it simply by Fourier methods, because the target grid is irregular. Separable filtering doesn’t work either, unless the irregular grid is separable as well. Therefore, to efficiently solve the problem in the general case, with any given desired precision, we used high-order B-spline interpolation with oversampling, as described by Unser [28]. Briefly, we approximate the sinc filter by the cardinal B-spline:

\[
\text{sinc} \approx \beta_m^{(k)} = \mathcal{F}^{-1}\left(\sum_{w=-\infty}^{\infty} \beta^{(k)}(w + 2\pi l)\right)
\]

where the $k$-th order B-spline is defined by the recursion $\beta^{(k)}(w) = \beta^{(0)} + \beta^{(k-1)}$, $\beta^{(0)} = \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}$. Note that this has the advantage that the filter can be decomposed in two parts. The denominator is a prefilter with infinite spatial support that is applied in the Fourier domain, where the image has finite support. Then the numerator, which has a small spatial support (of $k \times k$ pixels in the two-dimensional case), is applied in the spatial domain in a non-separable way, since it has to be evaluated on the irregular grid.

Furthermore, in order to improve accuracy with a minimal computational cost, we use a B-spline approximation of the sinc filter corresponding to $o$ times the original sampling rate (we can do so because $f$ and $f_n$ are band-limited), and we apply it to an $o$-times oversampled version of the image (times zero-padded in the Fourier domain). This improves accuracy because the B-spline approximation is very good except close to the Nyquist frequency. With oversampling, the part of the spectrum where the B-spline approximation is bad, lies outside the spectral support of the image.

From the spline interpolation error kernel in Unser’s article, we can compute the expected relative $L^2$ error when interpolating a band-limited function $f$ by $k$-th order B-spline interpolation with $o$-times oversampling, for different orders and oversampling factors. The graphic shows also the number of flops per pixel needed to interpolate for a given order and oversampling factor. Given this pre-computed chart, and a desired approximation precision $\delta$, we simply choose from it the
value of $k$ and $o$ which minimizes the computational cost and with expected error smaller than $\delta$.

4 A new reconstruction algorithm

The slow convergence of Gröchenig’s algorithm for our particular kind of data and the absence of a convergence theorem in this situation led us to the study of new algorithms. In particular we tried to combine: (i) the fact that we are in the perturbed sampling case (which is only exploited by the algorithms in section 3.1), with (ii) the fact that the perturbation is very smooth (which is only exploited by the algorithms in section 3.2), and (iii) the fact that the image itself is very regular, since it may have a fast decay near the Nyquist frequency, due to the instrument’s transfer function and higher sampling rates than conventional systems.

The algorithm is very similar to Aldroubi’s algorithm, it is based on alternating two operators, whose combination is a perturbation of the identity, and such that one of them simulates the original perturbed sampling. Only the choice of operators is different.

Observe that the irregular sampling process may be written in the following manner. Given a band-limited image $f$ (or equivalently its regular samples $s = \Delta_{\mathbb{Z}^2} \cdot f$, since we can reconstruct $f = \text{sinc} \ast s$), and the perturbation $\varepsilon$, consider the warping function $\phi_\varepsilon = I + \varepsilon$. Then the
irregular samples can be written as a regular sampling of the warped image
\[ \tilde{s} = \Delta_{Z^2} \cdot (f \circ \phi_\varepsilon) = \Delta_{Z^2} \cdot ((\text{sinc} * s) \circ \phi_\varepsilon) = A^+ s. \] (33)

We abbreviate this irregular sampling operator by \( A^+ \). Now all the problem is reduced to inverting the operator \( A^+ \). Its inverse is still a linear operator, but not necessarily a convolution-sampling operator. Nevertheless we can still approximate the inverse of \( A^+ \) by a convolution-sampling pair (that we shall call \( A^- \)) as follows: \(^4\)

\[ s_1 = \Delta_{Z^2} \cdot ((\text{sinc} * \tilde{s}) \circ \phi_\varepsilon^{-1}) = A^- s. \] (34)

This corresponds to performing the inverse warp from the irregular samples, seen as regular samples of the warped image \( \tilde{f} = f \circ \phi_\varepsilon \), by assuming that \( \tilde{f} \) is also band-limited to \([-\pi, \pi]^2\), which is not the case. In addition, some information has been lost because \( \tilde{f} \) has been sampled at the Nyquist rate corresponding to \([-\pi, \pi]^2\), while the spectrum of \( \tilde{f} \) is usually larger.

Note that in the case of constant \( \varepsilon \), both operators \( A^+ \) and \( A^- \) reduce to (opposite) translations by Shannon interpolation and one is the inverse of the other. Therefore, for very smooth “almost constant” \( \varepsilon \) we could expect \( A^- \) to be close to the inverse of \( A^+ \).

In the following we shall assume that \( A^- \) is a good approximation of the inverse of \( A^+ \) in the sense that
\[ A^- A^+ = (I - \alpha), \quad \text{with} \quad \|\alpha f\| \leq a\|f\|, \quad \text{and} \quad a < 1 \] (35)
for all \( f \) in a certain class. Then we have the equality
\[ s_1 = A^- A^+ s = (I - \alpha) s \] (36)

And \((I - \alpha)^{-1}\) can be expanded as a Neumann series, to recover \( s \) from \( s_1 \) by the following iteration
\[ s_n = s_1 + \alpha s_{n-1} = (I - \alpha^n) s \] (37)

so that
\[ \|s_n - s\| = \|\alpha^n s\| \leq a^n \|s\| \] (38)

Under the assumption given in equation (35), this means that the approximation error is reduced at each iteration by a factor \( a < 1 \), so the convergence is geometric.

In practice the operator \( \alpha \) is applied in two steps: First, a simulation \( \tilde{s}^{(n-1)} \) of the perturbed image from the current estimate \( s^{(n-1)} \) of the regular image:
\[ \tilde{s}^{(n-1)} = A^+ s^{(n-1)}; \] (39)

and secondly a correction of the errors found in this simulation with respect to the known perturbed image \( \tilde{s} \):
\[ s^{(n)} = s^{(n-1)} - A^- (\tilde{s}^{(n-1)} - \tilde{s}). \] (40)

### 4.1 Convergence analysis

Now we return to the conjecture in equation (35). It turns out that we cannot assume \( \|\alpha\| < 1 \) in general, but we may obtain \( \|\alpha f\| < \|f\| \) for all sufficiently regular \( f \), if \( \varepsilon \) is sufficiently small in amplitude:

\(^3\)Note that this operator can be efficiently calculated using the FFT, spatial convolutions and sampling, by the same oversampled B-spline procedure described at the end of the previous section.

\(^4\)Note that the computation of \( \phi_\varepsilon^{-1}(Z^2) \) can also be formulated as a perturbed sampling problem, but in this case, we largely have the necessary oversampling rate. Hence we can compute it very quickly using the ACT algorithm.
Proposition 2. Let \( f \in L^2(\mathbb{R}^2) \) with \( \text{supp}(\hat{f}) \subseteq [-\pi, \pi]^2 = R \), and let \( s = \Delta_{Z^2} \cdot f \) be the regular samples of \( f \), and \( \bar{s} = \Delta_{Z^2} \cdot (f \circ \phi_\varepsilon) \) its irregular samples, with a small perturbation \( \varepsilon \) with a much smaller spectral support \( \text{supp}(\varepsilon) \subset R \). Then the operator \( \alpha \) of the pseudo-inverse algorithm can be expanded up to third order as follows:

\[
\alpha s = \Delta_{Z^2} \cdot \left( (\mathcal{F}^{-1}( R) * D \langle Df, \varepsilon \rangle - D \text{alias}_R \langle Df, \varepsilon \rangle, \varepsilon) \right) + O(\varepsilon^3) \tag{41}
\]

The proof is given in appendix A, at the end of this chapter. Concerning the notation used in this proposition, observe that the aliasing operator associated to a sampling grid \( \Gamma \) (in our case \( \Gamma = Z^2 \)) or equivalently, to its Voronoi reciprocal cell \( R \) is

\[
(\text{alias}_R f)(x) = \mathcal{F}^{-1}( R) * \left( \sum_{k \in \Gamma^*} f(x)e^{i(k,x)} \right) \tag{42}
\]

or equivalently in the Fourier domain

\[
\mathcal{F}(\text{alias}_R f)(\omega) = \cdot R \cdot \left( \sum_{k \in \Gamma^*} \hat{f}(\omega + k) \right) \tag{43}
\]

i.e. the error that is incurred when sampling and sinc-interpolating a non-band-limited function \( f \).

Observe that \( \alpha s \) doesn’t have a linear term, it depends quadratically on \( \varepsilon \). From this expression we can also see that \( \|\alpha s\| \) depends essentially on the part of the spectrum of \( \langle Df, \varepsilon \rangle \) which goes beyond our spectral domain \( R \). But this can be controlled if both \( \varepsilon \) has a small spectral support, and \( f \) has relatively weak high frequencies.

Proposition 3. If \( \varepsilon \) is of the form given in equation (1) then

\[
\|\alpha s\|_2 \leq 4\pi^2 \|\varepsilon\|_{1,\infty} \left( \sum_{k=1}^{M} \nu_k \|A_k\|_1 (\|\hat{f}.B^+_k\|_2 + \|\hat{f}.B^-_k\|_2) \right) + O(\varepsilon^3) \tag{44}
\]

where \( \|A_k\|_1 \) denotes the sum of the amplitudes in the \( x \) and \( y \) directions, \( \|\varepsilon\|_{1,\infty} = \sup_{x \in \mathbb{R}^2} (|\varepsilon_1(x)| + |\varepsilon_2(x)|) \) is the maximal 1-norm of \( \varepsilon \) in the whole image, and \( \nu_k = (1 + \|\omega_k\|_{\infty}/(2\pi)) \) only varies between 1 (for low frequencies) and 1.5 (for high frequencies). Finally the spectral domain \( B^+_k \) is

\[
B^+_k = \{ \omega \in R : \exists \gamma \in 2\pi\mathbb{Z}^2, \gamma \neq 0, (\omega + \omega_k + \gamma) \in R \} \tag{45}
\]

the high frequency ring of width \( \omega_k \) within \( R = [-\pi, \pi]^2 \). Similarly \( B^-_k \) is a symmetrical set with respect to the origin.

If in addition we assume that \( f \) is sufficiently regular, in the sense that

\[
\|\hat{f}(w).B^+_k(w)\| \leq \frac{\text{area}(B^+_k)}{\text{area}([-\pi, \pi]^2)} \|\hat{f}\| \tag{46}
\]

then the upper bound (44) can be simplified to

\[
\|\alpha s\|_2 \leq 4\pi^2 \|\varepsilon\|_{1,\infty} \left( \sum_{k=1}^{M} \mu_k \|A_k\|_1 \right) + O(\varepsilon^3) \tag{47}
\]

where \( \mu_k = \left( 1 + \frac{\|\omega_k\|_{\infty}}{2\pi} \right) \frac{2\|\omega_k\|_{\infty}}{\pi} \) varies quadratically between 0 and 3, depending on the frequency \( \omega_k \).

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The proof is given in appendix A, at the end of this chapter. This proposition shows clearly that \( \alpha \) may in general be unbounded. The only way to ensure that \( \| \alpha f \| \leq a \| f \| \) with \( a < 1 \) is to ensure that \( f \) has relatively small high frequencies. This is usually the case for the initial image \( f \), because of the acquisition conditions. But it may not be the case for the iterates \( f_n \). Thus, a way to avoid the divergence of this algorithm is to precede each application of \( \alpha P \) which controls the high frequencies, for instance:

\[
P f = R_{R \setminus B_{k_0}}^* f, \quad \text{where} \quad k_0 = \arg \max_k \| \omega_k \|_{\infty}, \quad B_k = B_k^+ \cup B_k^-
\]  

To complete the analysis we consider the complementary projector

\[
Q f = \cdot B_{k_0}^* f
\]

which satisfies \( P + Q = I \) within the Payley-Wiener space. Now it is more reasonable to assume that \( \| \alpha P \| \leq a < 1 \). The modified iteration is:

\[
s_1 = A^- A^+ s &= (I - \alpha P) s - \alpha Q s \\
s_n = s_1 + \alpha P s_{n-1} &= (I - (\alpha P)^n) s - \left( \sum_{k=0}^{n-1} (\alpha P)^k \right) \alpha Q s
\]

so that

\[
\frac{\| s_n - s \|}{\| s \|} \leq a_n + \left( \frac{1 + a^n}{1 - a} \right) \frac{\| \alpha Q s \|}{\| s \|} 
\]

This equation shows that the modified iteration has a limit to accuracy which is determined by \( \| \alpha Q s \| \). This term will be small when the high frequencies \( Q s \) of \( s \) are relatively small, and \( \alpha \) does not excessively amplify these high frequencies, which is the case when \( \varepsilon \) is smooth too.

Now we return to the problem of showing that \( \| \alpha P \| \leq a < 1 \). The main idea is to try to deduce this from proposition 3. Now a major problem in applying it is that \( \varepsilon \) may have to be small, so that the unknown third order term \( O(\varepsilon^3) \) is smaller than a given constant \( c < 1 \). To solve this problem we have to apply this result locally, in such a way that the amplitude of \( \varepsilon \) is sufficiently reduced.

More precisely, assume that within any interval \( I \) of size smaller than \( L \) we have that \( \max_{x, y \in I} \| \varepsilon(x) - \varepsilon(y) \| \leq 2\delta \). Then we can consider \( \tilde{\varepsilon} = \varepsilon - \tilde{\varepsilon} \), which satisfies \( \| \tilde{\varepsilon} \| \leq \delta \) and hence in proposition 3, the third order term is smaller than \( c < 1 \). This analysis is valid because adding a constant to \( \varepsilon \) doesn’t change \( \alpha \), which can be easily verified.

Now we still have to translate a statement about a global upper bound of \( \| \alpha P \| \), into statements about upper bounds of \( \| \alpha P : I \| \) for intervals of size smaller than \( L \). We do so by considering a partition of unity:

\[
\sum_{k \in \mathbb{Z}^2} g_k = 1, \quad \text{with} \quad \text{supp}(g_k) \subseteq B_L(Lk) \quad (53)
\]

such as \( e.g. \) the rescaled first order B-splines \( g_k(x) = \beta^{(1)}(x/L - k) \). Then, since \( \alpha \) is linear we can write

\[
\| \alpha P f \|^2 = \| \sum_{k \in \mathbb{Z}^2} \alpha g_k P f \|^2 = \sum_{m, n \in \mathbb{Z}^2} \langle \alpha g_m P f, \alpha g_n P f \rangle 
\]

Suppose that we can bound this term by

\[
\sum_{m, n \in \mathbb{Z}^2} \langle \alpha g_m P f, \alpha g_n P f \rangle \leq \sum_{m, n \in \mathbb{Z}^2} b_{n,m} \| g_m P f \| \| g_n P f \| 
\]

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where for all \( n \), we have \( \sum_{m=-\infty}^{\infty} |b_{n,m}| \leq a^2 < 1 \). If this conjecture holds, then we have
\[
\|\alpha P f\|^2 \leq y^T B y
\]  
(56)
where \( B = (\{|b_{n,m}|\}) \) is the matrix composed of the coefficients \( b_{n,m} \) in equation (55), and the vector \( y \) has coordinates \( y_n = \|g_n P f\| \). Using Gerschgorin’s theorem [10], we deduce that the eigenvalues of \( B \) lie within the union of the circles \( D_n \), plus the limits \( D_{\pm\infty} \), where each \( D_n \) is centered at \( |b_{n,n}| \) and has radius \( r_n = \sum_{m \neq n} |b_{m,n}| \). Hence all the eigenvalues of \( B \) are smaller than \( \sum_{m=-\infty}^{\infty} |b_{n,m}| \leq a^2 < 1 \). In conclusion \( y^T B y \leq a^2 \|y\|^2 \), i.e.
\[
\|\alpha P f\|^2 \leq a^2 \sum_n \|g_n P f\|^2 = a^2 \int_{\mathbb{R}^2} |P f|^2 \left( \sum_n g_n^2 \right)
\]  
(57)
But since \( g_n \) is a partition of unity with \( g_n \in [0,1] \), we have that \( g_n^2 \leq g_n \) and hence \( \sum_n g_n^2 \leq 1 \). In fact this sum is a function which oscillates between 1 and 0.5. Finally, the projection operator can only make decrease the norm of \( f \), and we have:
\[
\|\alpha P f\|^2 \leq a^2 \int_{\mathbb{R}^2} |P f|^2 = a^2 \|P f\|^2 \leq a^2 \|f\|^2.
\]  
(58)
In order to complete this proof we still need to show conjecture (55) and provide a criterion to compute the maximal size \( L \) of a neighborhood, within which \( \varepsilon \) is sufficiently small, for the third order term \( O(\varepsilon^3) \) in proposition 3 to be smaller than a given constant \( c < 1 \).

### 4.2 Numerical approximation

When applying the operator \( \alpha \) in equation (37) or the operator \( \alpha P \) in the modified algorithm in equation (51) the operators \( A^+ \) and \( A^- \) will be approximated in our algorithm by \( \alpha_\delta = A_\delta^- A_\delta^+ - I \), whereas in equation (36) it will be approximated by \( \alpha_\delta' = A_\delta^- A^+ - I \), since the first application if \( A^+ \) is performed by the acquisition system, not by our algorithm. It is easy to show that \( \lim_{\delta \to 0} \alpha_\delta = \lim_{\delta \to 0} \alpha_\delta' = \alpha \) where \( \delta = 1/\sigma \) for the damped sinc approximation, and \( \delta = 1/m \) for the cardinal B-spline approximation. Hence for a sufficiently small \( \delta \) we shall have \( \|\alpha_\delta\| < a < 1 \) and \( \|\alpha_\delta'\| < a < 1 \). To summarize, the iterations with the numerical approximation of the sinc filter are as follows:

\[
s_\delta^{(1)} = (I - \alpha_\delta') s
\]  
(59)
\[
s_\delta^{(n)} = \alpha_\delta \cdot s_\delta^{(n-1)} + s_\delta^{(1)}
\]  
(60)
\[
= \left( I + \sum_{i=1}^{n-2} \alpha_\delta^i \right) (\alpha_\delta - \alpha_\delta') - \alpha_\delta^{n-1} \alpha_\delta' \cdot s
\]

Observe that the relative error \( \|s_\delta^{(n)} - s\|/\|s\| \) still has a term \( \|\alpha_\delta^{n-1} \alpha_\delta'\| < a^n \) that decreases geometrically with the iterations, but a second term appears which may increase with the iterations but is bounded by \( \frac{a^n}{n!} \|\alpha_\delta - \alpha_\delta'\| \) (provided \( \|\alpha\| \leq a < 1 \)). This second term can be kept below the noise level as long as \( a < 1 \) and \( \delta \) is sufficiently small.

\[\text{Observe that for } n = m \text{ the value of } b_{n,m} \text{ is provided by proposition 3 and the previous discussion on the local application of this proposition. In the case } |n-m| > 1, \text{ the two window functions } g_n \text{ and } g_m \text{ have no common support, and the only reason for } (\alpha g_n P f, \alpha g_n P f) \text{ to be non-zero is that } \alpha f \text{ expands the spatial support of } f, \text{ due to the sinc filters it contains. Finally, the case } |n-m| = 1 \text{ may require a stationarity hypothesis on } f \text{ in order to show that for a given } n, \sum_{|n-m|=1} |b_{n,m}| \leq 8|b_{n,m}|.\]

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5 Experiments

In order to test and compare the performance of the different algorithms we constructed a set of simulated satellite images. We started from a very high resolution aerial image $g$ which was later filtered with the transfer function of different imaging systems, particularly a CCD array with sharp optics, and the supermode and hypermode systems (see [23, 26] and annex B). In all cases the transfer function was truncated beyond the corresponding Voronoi cell so that aliasing effects were not taken into account, and we obtained perfectly band-limited images. The application of the transfer function is done simply to simulate the spectral contents of images obtained with different instruments. This produced the reference image $s$ with spectral support $[-\pi, \pi]^2$.

Then we simulated different perturbations $\varepsilon$ such that $\text{supp}(\varepsilon) \subseteq [-\frac{\pi}{T}, \frac{\pi}{T}]^2$ for different values of $T > 1$, and for different standard deviations for $\varepsilon(x)$ (typically 0.25, 0.5 and 1). With the given perturbation we simulated the perturbed images $\tilde{s}$ with a high precision (usually $10^{-8}$) using the oversampled B-spline interpolation technique described at the end of section 3.2 (Aldroubi’s algorithm). Finally we added some white noise to the irregular samples with standard deviation $10^{-3}$ times smaller than the standard deviation of the image ($SNR = 60dB = 10^3$). Then we asked the three methods to recover $s$ up to a precision of $10^{-3}$.

Concerning the pseudo-inverse method, we observed that the stabilized version (the one in equations (50) and (51) which applies $\alpha P$ at each iteration), even if it avoids divergence, it converges with an accuracy which is less precise than the one obtained by intermediate iterates of the divergent version in equations (36) and (37). Furthermore, the point at which it starts to diverge can be detected (without knowledge of $s$ of course), at the point where the residual stops decreasing. This usually happens one or two iterations after the actual error stops decreasing. For this reason we always applied the non-convergent version, because it usually provides a better accuracy, and the divergence point can be determined. In the graphs, however, we show many more iterations beyond the detection of the divergence point (for all algorithms), in order to show its behaviour.

Figures 2 to 5 show some representative results. The experiments confirm the influence of the regularity of the image and of the the amplitude and the regularity of the perturbation in the convergence of the pseudo-inverse algorithm. They also show how the ACT algorithm, usually provides the finest accuracy, but at the cost of more computations. The pseudo-inverse algorithm, on the other hand, is the fastest to provide a good approximation, especially for regular images and perturbations, even though its precision may be less fine than that of the other two methods if the image is very sharp. In many cases the computational effort of the pseudo-inverse with respect to the ACT algorithm to reach the same accuracy is a factor of 3, 5 or even 10 times smaller. Finally, Aldroubi’s method presents an intermediate behaviour, often producing an approximation level just between those of the pseudo-inverse and the ACT algorithms, and also with an intermediate computational cost.

Further experiments on the pseudo-inverse algorithm confirm the convergence properties described in section 4.1. For instance, if the image has a spectral contents similar to Hipermode, the perturbation consists of a single sinusoidal oscillation with amplitude $A$ and period $T$, then the minimal reconstruction error $\|f - f_n\|$ during the pseudo-inverse iterations was at least 5 times smaller than the initial error $\|f - \hat{f}\|$ whenever $\frac{A}{T} < 0.1$. Similarly, if the perturbation was a colored noise with standard deviation $A$ (maximal perturbation is more than 2.5A) and spectrum in $[-\frac{2\pi}{T}, \frac{2\pi}{T}]^2$, then we obtained a reconstruction error 5 times smaller than the initial error whenever $\frac{A}{T} < 0.05$. This means that if the image is sufficiently smooth, for perturbations with frequencies smaller than $\frac{\pi}{10}$ for instance we can still obtain a reasonable reconstruction with perturbation amplitudes up to about 2 pixels.
Figure 2: *The influence of the spectral contents of the image* on the performance of the methods. Observe that for images with weak high frequencies such as hypermode, the pseudo-inverse method reaches the attainable precision much faster than the other two methods. For sharper images, however, the pseudo-inverse attains a less precise accuracy than the other two methods. In all cases the method of Gröchenig, Strohmer and Rauth attains the most precise accuracy faster, but intermediate precisions may be attained faster by Aldroubi’s method.

In all cases, the perturbation is a colored noise with standard deviation of 0.5 pixels (maximal perturbation is 1.25 pixels), and spectral contents inside \([-\frac{T}{2}, \frac{T}{2}]^2\) for \(T = 20\), and the SNR of the sampled image is 60 dB, *i.e.* the \(L^2\) norm of the noise is \(10^{-3}\) times smaller than the \(L^2\) norm of the image. The approximation level of the oversampled B-spline filters was fixed at the same level.
Figure 3: The influence of the spectral contents of the perturbation on the performance of the methods. Observe that when the frequency of the perturbation increases the performance of the proposed pseudo-inverse method is affected to a larger extent than the other methods. In both cases, the spectral contents of the image is similar to a CCD with good optics. For less sharp images like supermode or hypermode the degradation of the performance of the pseudo-inverse is less important.

In both cases the SNR of the sampled image is 60 dB, \( i.e. \) the \( L^2 \) norm of the noise is \( 10^{-3} \) times smaller than the \( L^2 \) norm of the image. The approximation level of the oversampled B-spline filters was fixed at the same level. The perturbation is a colored noise with standard deviation of 0.5 pixels (maximal perturbation is 1.25 pixels), and spectral contents inside \( [-\frac{\pi}{T}, \frac{\pi}{T}] \) for \( T = 20 \) (subfigure a), or \( T = 10 \) (subfigure b).

Table 2: Accuracy test of the pseudo-inverse algorithm. In this experiment the reference image \( f \) has spectral contents similar to Hipermode. The perturbation \( \varepsilon \) is either a purely horizontal oscillation \( A \sin(2\pi x/T) \) (sin column) or its Fourier transform is a white noise with standard deviation \( A \) inside \( [-\frac{\pi}{T}, \frac{\pi}{T}] \) and zero outside this cell (rand column). Then starting from the perturbed image \( \tilde{f} \), for a given value of \( T \) and \( A \), we computed the pseudo-inverse reconstruction to obtain a sequence \( f_n \) of approximations to the reference \( f \), and kept the iteration \( n \) that minimizes the reconstruction error \( \|f_n - f\| \). To the left of each column we show \( \|f_n - f\| \) (the improvement with respect to the original), and to the right of each column we show \( \frac{\|f_n - f\|}{\delta} \) where \( \delta = 10^{-3} \) is the approximation level used to chose appropriate B-spline filters.
Figure 4: The influence of the amplitude of the perturbation on the performance of the methods. Observe that the attainable accuracy is affected for all methods by this amplitude. In all cases, the spectral contents of the image is similar to supermode and the SNR of the sampled image is 60 dB, i.e. the $L^2$ norm of the noise is $10^{-3}$ times smaller than the $L^2$ norm of the image. The approximation level of the oversampled B-spline filters was fixed at the same level. The perturbation is a colored noise with standard deviation of 0.5 pixels (maximal perturbation is 1.25 pixels), and spectral contents inside $[-\frac{\pi}{T}, \frac{\pi}{T}]^2$ for $T = 20$. 
Figure 5: The influence of noise on the performance of the methods. The conditions of this experiment are exactly the same as those in figure 4, except that we added no noise. Observe how noise only acts as a limit to the attainable accuracy (at $10^{-3}$ in figure 4), without any further effects.
6 Conclusions and perspectives

We reviewed the theory of irregular sampling and the available algorithms and discussed its application to satellite imaging. Unfortunately none of the results on existence of a stable reconstruction formula do exactly apply to our case. Kadec-type results are too restrictive in terms of the allowed amplitude, whereas Beurling-Landau-type results are too restrictive in terms of the spectral support of the image. However Kadec’s results do not make use of the regularity of the image\(^6\), nor of the regularity (the small spectral support) of the perturbation. On the other hand the results of Beurling and Landau do exploit the regularity of the perturbation in some way (the density condition can be translated in the case of perturbed sampling in terms of a bound on \(D\varepsilon\)), but not the regularity of the image, and the fact that the sampling is perturbed (not just a general irregular sampling). The question of whether these particular properties could eventually be used to relax Kadec’s or Beurling’s results is to the best of our knowledge still open.

Concerning reconstruction algorithms, none of the algorithms we analyzed provides a convergence result in the conditions we need to apply it. Nevertheless two of them, and the new pseudo-inverse algorithm we propose here, do provide reasonable approximations, below the noise level that usually occurs in satellite imagery.

It seems that our problem requires a shift in the formulation of the problem. We do no longer look for the conditions under which exact stable reconstruction is possible (this may not be possible, even if we take into account the regularity of the image and the perturbation), but for conditions under which the problem of reconstructing the regular samples \(up to a certain accuracy\) is reasonably well posed, and for criteria to determine which is the best possible \(attainable accuracy\) given the characteristics of the perturbation.

From an engineering point of view the choice between the three methods we outlined here depends on many factors, ranging from the importance of speed vs. accuracy, to the acquisition conditions. Under certain conditions, the pseudo-inverse produces results with the same accuracy as the ACT method, with less than 5 times less computations. In other conditions, the accuracy of the pseudo-inverse may be 10 times poorer than that of the ACT method.

We did not address here possible aliasing problems (or its equivalent in the irregular sampling case). The function was assumed to be perfectly band-limited. The instrument’s transfer function was used only within \([-\pi, \pi]^2\) in order to simulate the kind of decay rate that we get from the sampling instrument. But beyond \([-\pi, \pi]^2\) either \(f\) or the transfer function were assumed to be zero, which is not strictly true.

A more accurate approach would be to assume \(f\) inside \(V^p(H)\) for the known transfer function \(H\). But computing orthogonal projections to \(V^p(H)\) can be computationally very expensive. Alternatively we can use Aldroubi’s algorithm to find the best approximation \(\tilde{f} \in V^p(\phi)\) to the real \(f\) (given the irregular samples \(f(\Lambda)\)) by using a convenient, \(e.g.\) spline-like, generator \(\phi\) which allows for fast computation, and at the same time is close to \(H\). Note that in such a case the correct projector \(P\) in Aldroubi’s algorithm should be a projection from \(L^p\) into \(V^p(\phi)\) which is as close as possible to \(V^p(H)\). This can be done by the oblique projection \(P_{V^p(\phi) \perp V^p(H)}\) as suggested by Aldroubi, Feichtinger and Unser [1, 28]. The only difference from the computational point of view is the calculation of the prefilter, which can be done with the same cost in the Fourier domain.

\(^6\)in the sense that the image is not only band-limited but its Fourier coefficients also are small close to the Nyquist frequency
7 Acknowledgement

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A Proofs of convergence analysis.

In the proof of propositions 2 and 3 we use several times the following two results

**Lemma 1.** If a function \( f : \mathbb{R}^2 \to \mathbb{R} \) has spectral support \( \text{supp}(\hat{f}) \subseteq R \) in a rectangular region \( R = [-r_1, r_1] \times [-r_2, r_2] \), then
\[
\|Df\|_{L^2}^2 \leq (r_1^2 + r_2^2)\|f\|_{L^2}^2
\]
and therefore \( Df = O(f) \).

This result follows simply from Parseval’s formula and the fact that \( \mathcal{F}(Df)(w) = w \hat{f} \). We shall apply it both to bound the norm of the image \( f \) and the perturbation \( \varepsilon \).

The next result provides a convenient development of the inverse warp \( \phi_\varepsilon^{-1} \) as a truncated power series in \( \varepsilon \). (Recall that \( \phi_\varepsilon^{-1} \) is the inverse of the warp \( \phi_\varepsilon(x) = x + \varepsilon(x) \) mapping the regular grid \( \mathbb{Z}^2 \) to the irregular grid \( \Lambda \).

**Lemma 2.** The inverse warp \( \phi_\varepsilon^{-1} \) can be expanded as
\[
\phi_\varepsilon^{-1}(x) = x - \varepsilon_1(x) + O(\varepsilon^3) \quad \text{where} \quad \varepsilon_1(x) = \varepsilon(x - \varepsilon(x))
\]

**Proof.** Let’s consider the approximation \( \tilde{\phi}_\varepsilon^{-1} = x - \varepsilon_1(x) \) and test how it approximates the inverse of \( \phi_\varepsilon(x) = x + \varepsilon(x) \):
\[
\tilde{\phi}_\varepsilon^{-1}(\phi_\varepsilon(x)) = x + \varepsilon(x) - \varepsilon(x + \varepsilon(x)) - \varepsilon(x + \varepsilon(x)) = x - D\varepsilon(x) \cdot d(x) - D^2\varepsilon(d(x), d(x)) + O(\varepsilon^3).
\]

In the last equation we used the abbreviation \( d(x) = \varepsilon(x) - \varepsilon(x + \varepsilon(x)) \), which can itself be expanded as
\[
d(x) = D\varepsilon(x) \cdot \varepsilon(x) + O(\varepsilon^2).
\]

Using lemma 1 we conclude that \( d(x) = O(\varepsilon^2) \), \( D^2\varepsilon = O(\varepsilon) \) so that the second order term in equation (63) is actually \( O(\varepsilon^5) \) and is absorbed in \( O(\varepsilon^3) \). Furthermore the first order term \( D\varepsilon(x) \cdot d(x) \) is actually \( O(\varepsilon^3) \), because \( D\varepsilon = O(\varepsilon) \) using once more lemma 1. So equation (63) can be rewritten as
\[
\tilde{\phi}_\varepsilon^{-1}(\phi_\varepsilon(x)) = x + O(\varepsilon^3) = \phi_\varepsilon^{-1}(\phi_\varepsilon(x)) + O(\varepsilon^3).
\]

**Proof of proposition 2.** We start by writing step by step the application of the operator \( I - \alpha = A^- A^+ \) to an image \( s = \Delta_{\Gamma} f \) such that \( \text{supp}(\hat{f}) \subseteq R \). (Recall that the sampling grid is \( \Gamma = \mathbb{Z}^2 \) and \( R = [-\pi, \pi]^2 \) is the corresponding reciprocal cell, so that the Shannon interpolation filter is \( \mathcal{F}^{-1}(R) = \text{sinc} \). We shall note by \( \text{sinc}^c = \mathcal{F}^{-1}(R^c) \) the complementary filter. Note that \( \text{sinc}^c * f = f - \text{sinc} * f \).

\[
\begin{align*}
  f &= \text{sinc} * s & s &= \Delta_{\Gamma} f \\
  \hat{f} &= f \circ \phi_\varepsilon & \hat{s} &= \Delta_{\Gamma} \cdot \hat{f} = A^+ s
\end{align*}
\]

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Since the spectral support of \( \hat{f} \) goes beyond \( R \), it cannot be recovered from \( \hat{s} \). However, the interpolation of \( \hat{s} \) yields a certain function \( \hat{f}' \) with spectral support in \( R \), that coincides with \( \hat{f} \) at the sampling points

\[
\hat{f}' = \text{sinc} * \hat{s} = \text{sinc} * \hat{f} + \text{alias}(\hat{f}) \quad \hat{s} = \Delta \Gamma \cdot \hat{f}'
\]  

(68)

Next we apply \( A^{-} \hat{s} = \Delta \Gamma \cdot ((\text{sinc} * \hat{s}) \circ \phi_{\epsilon}^{-1}) \) in three steps: convolution (we just did it as \( \hat{f}' = \text{sinc} * \hat{s} \)), inverse warp, and sampling

\[
f_{1}' = \hat{f}' \circ \phi_{\epsilon}^{-1} \quad s_{1} = \Delta \Gamma \cdot f_{1}' = A^{-} \hat{s}
\]  

(69)

Finally consider the interpolation of \( s_{1} \):

\[
f_{1} = \text{sinc} * s_{1} = \text{sinc} * f_{1}' + \text{alias}(f_{1}') \quad s_{1} = \Delta \Gamma \cdot f_{1} = A^{-} \hat{s}
\]  

(70)

In the sequel we shall use the abbreviation

\[
S(f) = \text{sinc} * (\Delta \Gamma \cdot f) = \text{sinc} \cdot f + \text{alias}_{R}(f)
\]  

(71)

for the sampling-interpolation pair of operators (of course \( S \) is the identity if \( \text{supp}(\hat{f}) \subseteq R \)). With this notation the last equation becomes \( f_{1} = S(f_{1}') \).

Now we are ready to compute the Taylor development of \( \alpha f = f_{1} - f \) with respect to \( \epsilon \). To do so we start by computing the third order Taylor development of \( \hat{f} \) and \( \hat{f}' \).

\[
\hat{f} = f + Df \cdot \epsilon + \frac{1}{2} D^2 f(\epsilon, \epsilon) + O(\epsilon^3)
\]  

(72)

\[
\hat{f}' = f + \text{sinc} * (Df \cdot \epsilon) + \frac{1}{2} \text{sinc} * (D^2 f(\epsilon, \epsilon))
\]  

\[+ \text{alias}_{R}(Df \cdot \epsilon) + \frac{1}{2} \text{alias}_{R}(D^2 f(\epsilon, \epsilon)) + O(\epsilon^3)
\]  

(73)

Since according lemma 2, the inverse warp is \( \phi_{\epsilon}^{-1} = x - \epsilon_{1}(x) + O(\epsilon^3) \), the Taylor development of \( f_{1}' \) can be written

\[
f_{1}' = \hat{f}' - D\hat{f}' \cdot \epsilon_{1} + \frac{1}{2} D^2 f(\epsilon_{1}, \epsilon_{1}) + O(\epsilon^3).
\]  

(74)

In order to express this development in terms of \( f \) and \( \epsilon \), we first expand \( \epsilon_{1}(x) \) as

\[
\epsilon_{1}(x) = \epsilon(x - \epsilon(x)) = \epsilon(x) - D\epsilon(x) \cdot \epsilon(x) + O(\epsilon^3)
\]  

(75)

\[= \epsilon(x) + O(\epsilon^2)
\]  

(76)

and secondly, we expand \( D\hat{f}' \) and \( D^2 \hat{f}' \) up to the right order using equation (73)

\[
D\hat{f}' = Df + \text{sinc} * (D^2 f(\epsilon, \cdot) + Df \cdot D\epsilon) + D \text{alias}_{R}(Df \cdot \epsilon)
\]  

(77)

\[+ O(\epsilon^2)
\]  

\[
D^2 \hat{f}' = D^2 f + O(\epsilon).
\]  

(78)

Substituting the last three equations in equation (74) we obtain

\[
\begin{align*}
 f_{1}' &= f + \text{sinc} * (Df \cdot \epsilon) + \text{sinc} * (D^2 f(\epsilon, \epsilon)) + \text{alias}_{R}(Df \cdot \epsilon) + \text{alias}_{R}(D^2 f(\epsilon, \epsilon)) \\
&= \hat{f}' + O(\epsilon^3)
\end{align*}
\]  

(79)
Sampling and reinterpolation of \( f'_1 \) introduces new aliases in all terms except for \( \hat{f}' \) whose spectral support is already in \( \mathbb{R} \):

\[
\begin{align*}
  f_1 &= f + \text{sinc} \ast (Df \cdot \varepsilon) + \text{alias}_R(Df \cdot \varepsilon, \varepsilon_1) + \frac{1}{2} \text{sinc} \ast (D^2f(\varepsilon, \varepsilon)) + \text{alias}_R(D^2f(\varepsilon, \varepsilon)) \\
  &= \hat{f}' + O(\varepsilon^3) \\
  \intertext{Finally we use the expression of \( \varepsilon_1 \) given in equation (75) in order to group the terms}
  0 &= \text{sinc} \ast \left[ (Df \cdot \varepsilon) - (Df \cdot (\varepsilon - D\varepsilon \cdot \varepsilon + O(\varepsilon^3))) - ((\text{sinc} \ast (Df \cdot D\varepsilon)) \cdot (\varepsilon + O(\varepsilon^2))) \right] \\
  &= \text{sinc} \ast \left[ \text{sinc}^c \ast [Df \cdot D\varepsilon] \cdot \varepsilon + O(\varepsilon^3) \right] \\
  0 &= \text{sinc} \ast \left[ \frac{1}{2} D^2f(\varepsilon, \varepsilon) + \frac{1}{2} Df(\varepsilon + O(\varepsilon^3), \varepsilon + O(\varepsilon^3)) - ((\text{sinc} \ast D^2f(\varepsilon, \cdot)) \cdot (\varepsilon + O(\varepsilon^2))) \right] \\
  &= \text{sinc} \ast \left[ \text{sinc}^c \ast [D^2f(\varepsilon, \cdot)] \cdot \varepsilon + O(\varepsilon^3) \right]
\end{align*}
\]  

Similarly we conclude that

\[
\begin{align*}
  0 &= \text{alias}_R(\text{sinc}^c \ast [Df \cdot D\varepsilon] \cdot \varepsilon) + O(\varepsilon^3) \\
  0 &= \text{alias}_R(\text{sinc}^c \ast [D^2f(\varepsilon, \cdot)] \cdot \varepsilon) + O(\varepsilon^3)
\end{align*}
\]  

Taking into account that \( D(Df \cdot \varepsilon) = D^2f(\varepsilon, \cdot) + Df \cdot D\varepsilon \), and recalling the notation \( S(g) = \text{sinc}^c \ast g + \text{alias}_R(g) \) for the sampling-interpolation pair, we can summarize equation (80) as follows:

\[
f'_1 = f + S([\text{sinc}^c \ast (D(Df \cdot \varepsilon)) - D\text{alias}_R(Df \cdot \varepsilon)] \cdot \varepsilon) + O(\varepsilon^3)
\]  

Then, equation (41) follows by sampling once more (since sampling after applying \( S \) is equivalent to just sampling).

\[\square\]

\textbf{Proof of proposition 3.} We abbreviate the main term in equation (85) as

\[
C_\varepsilon f = S([\text{sinc}^c \ast (D(Df \cdot \varepsilon)) - D\text{alias}_R(Df \cdot \varepsilon)] \cdot \varepsilon)
\]  

and we further decompose both components of the vector \( C_\varepsilon = \begin{pmatrix} C_{1,\varepsilon} \\ C_{2,\varepsilon} \end{pmatrix} \), as well as \( \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \) and \( D = (D_1, D_2) \):

\[
C_{j,\varepsilon} f = S \left( \text{sinc}^c \ast (D_j(D_1f \varepsilon_1 + D_2f \varepsilon_2)) - D_j\text{alias}_R^j(D_1f \varepsilon_1 + D_2f \varepsilon_2) \right) + o(\varepsilon^3) \quad \text{for } j = 1, 2
\]  

This way \( \alpha f \) can be written as

\[
\alpha f = C_\varepsilon f \cdot \varepsilon + o(\varepsilon^3)
\]

\[
= C_{1,\varepsilon} f \varepsilon_1 + C_{2,\varepsilon} f \varepsilon_2 + o(\varepsilon^3)
\]
To find an upper bound of \( \| \alpha f \|_{L^2} \) we apply first the triangular inequality to the scalar product \( D_\varepsilon f \cdot \varepsilon \), and then we compute the max of \( \varepsilon_j \) over the whole domain:
\[
\| \alpha f \|_2 \leq \| C_1,\varepsilon f \varepsilon_1 \|_2 + \| C_2,\varepsilon f \varepsilon_2 \|_2 + o(\varepsilon^3)
\]
\[
\leq \| C_1,\varepsilon f \|_2 \| \varepsilon_1 \|_\infty + \| C_2,\varepsilon f \|_2 \| \varepsilon_2 \|_\infty + o(\varepsilon^3)
\]
(90) (91) (92)

(Observe that we could as well have applied Cauchy-Schwartz, but this would introduce a factor in \( \| \varepsilon \|_{L^2} \) which depends on the image size). Next, we apply once again the triangular inequality, this time to \( \| C_j,\varepsilon f \|_2 \):
\[
\| C_j,\varepsilon f \|_2 \leq \| \sin^c * D_j (D_1 f \varepsilon_1) \|_2 + \| D_j \text{alias}_R (D_1 f \varepsilon_1) \|_2 + \| \sin^c * D_j (D_2 f \varepsilon_2) \|_2 + \| D_j \text{alias}_D (D_2 f \varepsilon_2) \|_2 + o(\varepsilon^3)
\]
(93)

Finally, each of the four terms (for \( j = 1 \)) can be upper bounded, by taking into account the link between differentiation in the Fourier domain, the aliases and the different spectral supports:
\[
\| \sin^c * D_1 (D_1 f \varepsilon_1) \|_2 \leq \sum_{k=1}^N |A_{1,k}| \left( \| \hat{f} \cdot B_k^+ \| + \| \hat{f} \cdot B_k^- \| \right) \pi^2
\]
(94)
\[
\| \sin^c * D_1 (D_2 f \varepsilon_2) \|_2 \leq \sum_{k=1}^N |A_{2,k}| \left( \| \hat{f} \cdot B_k^+ \| + \| \hat{f} \cdot B_k^- \| \right) \pi (\pi + |w_{1,k}|)
\]
(95)
\[
\| D_1 \text{alias}_D (D_1 f \varepsilon_1) \|_2 \leq \sum_{k=1}^N |A_{1,k}| \left( \| \hat{f} \cdot B_k^+ \| + \| \hat{f} \cdot B_k^- \| \right) (\pi - \frac{1}{2}|w_{1,k}|)^2
\]
(96)
\[
\| D_1 \text{alias}_D (D_2 f \varepsilon_2) \|_2 \leq \sum_{k=1}^N |A_{2,k}| \left( \| \hat{f} \cdot B_k^+ \| + \| \hat{f} \cdot B_k^- \| \right) \pi^2
\]
(97)

A similar upper bound can be found for \( j = 2 \), and by substituting \( |w_{1,k}| \) and \( |w_{2,k}| \) by the common upper bound \( |w_k|_\infty \) (the largest coordinate in module), we obtain (44), after simply rearranging the terms. This completes the proof of the first part of the proposition.

To show the second part we just observe that the upper bound \( \frac{\text{area}(B_k^+)}{\text{area}([-\pi, \pi]^2)} \| \hat{f} \| \) for \( \| \hat{f} \cdot B_k^+ \| \), can be itself upper-bounded by:
\[
\frac{\text{area}(B_k^+)}{\text{area}([-\pi, \pi]^2)} = (\lambda_1 + \lambda_2) - \lambda_1 \lambda_2 \leq 2 \sup \{ \lambda_1, \lambda_2 \} = \frac{2|w_k|_\infty}{\pi}
\]
(98)
where \( \lambda_j = \frac{|w_{j,k}|}{2\pi} \).

\[\square\]

**Sample application** As an application of the upper bound provided by proposition 3, assume that there \( N \) vibrating modes, with amplitude \( |A_{j,k}| < A \) and frequency \( |w_k|_\infty < \frac{\pi}{T} \). Then, \( \| \varepsilon \|_\infty < NA \), so
\[
\frac{\| \alpha f \|_2}{\| f \|_2} \leq (4NA)^2 4\pi^2 \frac{2}{T} (1 + \frac{1}{2T})
\]
To have convergence in the pseudo-inverse algorithm, this value should be smaller than 1 (minus the term in \( O(\varepsilon^3) \)), i.e.:
\[
A \leq \frac{1}{8N\pi \sqrt{\frac{2}{T} (1 + \frac{1}{2T})}}
\]

For instance, for \( T = 10 \) we may obtain convergence for \( A \leq 0.18 \frac{1}{N} \). For a single vibrating mode (\( N = 1 \)) this means that the maximal perturbation amplitude \( 2A \) of the sinusoidal functions shouldn’t exceed 0.36 pixels, which is already larger than Kadec’s result. However, since we do not control the third order term, this does not ensure convergence.

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Perspectives. The upper bound for \( \|\alpha s\| \) we just showed is not very tight, and can be improved in several ways.

First the regularity hypothesis we assume on \( f \) (white noise) is very weak. Natural images are usually much more regular, with a faster decay of Fourier coefficients. If we take into account this decay and the decay imposed by the transfer function, we can obtain a tighter upper bound. Secondly, our upper bound considers the worst case, which is rarely attained, especially when \( \varepsilon \) contains many Fourier coefficients. If we consider mean errors (by taking expected values instead of sup’s, the upper bounds would be still smaller. For instance the expected value of \( \|\varepsilon_j\|_\infty \) for \( \varepsilon_j \) with \( N \) Fourier coefficients having amplitude \( \leq A \), but a random phase, is \( \sqrt{NA} \), which is considerably smaller than the preceding upper-bound \( NA \).

Finally, this upper bound should be applied locally as explained in the main text. Since this allows to reduce the amplitude (to the local variations, not taking into account very low-frequency components of the perturbation), as well as the number \( N \) of terms defining \( \varepsilon \), since only a few are significant in a small domain.

Références


